A Study on Musical Sieves

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I Introduction

A musical sieve is a tool used for organizing musical information pioneered by Iannis Xenakis. In this paper I will first consider a general musical form, which will motivate a much heavier mathematical discussion on sieves. I will then return to considering musicality informed both by concrete aesthetics and abstract organization.

II The quality of music

When I speak of 'music', I am referring to *considered sound* - sound that is interpreted in a coherent form. This may seem too general, too subjective, to support formal mathematical structures, but that is far from the truth. As demonstrated in our very class, there exist strong tendencies across many different cultures to deliberately structure their music in logically ways. The necessity for coherency is a part of our desire for understanding - and music itself is formed in the crucible of rigid structure and curiosity for the unknown.

Logic, like language, and most semiotic sign systems, is discrete. Some cognitive scientists, like Jerry Fodor, even argue that discreteness is one of the factors that makes human thought unique, a necessary precondition for advanced reason. But sound is continuous! It is not surprising then sound must undergo a process of discretization to be intelligible to us. This discretization happens in many forms: pitches, rhythms, dynamics, timbres, attacks - all of them have common discretizations. These parameters can sometimes vary continuously, such as in crescendos or slides, but these continuous variations are most often thought of and written as discrete musical events themselves. Intervals between discrete events create new spaces of continuity, which can then be further discretized. Even in music that deliberately eschew discretness, such as in my modern and contemporary trombone works, it is impossible to escape the discreteness of time and form - that all works must have a beginning, middle and end. In table 1, I have a list of musical parameters and their associated discretizations.

Intervals provide one way to consider continuity within discrete parameters. Another common way comes from performance and probability. Performance itself is a continuous act, and we often bracket this continuity as 'interpretation' - the act of realizing and abstract discrete piece of music as continuous sound. Even speakers must do this - as no two speakers sound the same. Composers can use this sort of continuity within the act of composition itself via probability theory.

Parameter	Discretization
Pitch	Notes, Fractions
Loudness	Dynamic Markings, Volume
Attack	Accents, Dynamic Markings
Timbre	Instrumentation, Harmonics
Meter	Time Signature
Rhythm	Noteheads
Form	Section, Movement Number
Location	Orchestration, Speaker Orientation
Intervallic Discretizations	
Pitch	Glissando, Vibrato
Loudness	Crescendo, Tremolo
Rhythm	Rubato, Swing
Location	Panning, Movement
Timbre	"Shape" (No settled terminology)
Form	Fade Out/In

Table 1: List of parameters and their discretizations

Probability theory allows us to make continuity intelligible within our discrete logical world. Every discretization above can also be mediated probabalistically to move closer towards the continuous world of sound. Music is an abstraction, and therefore necessarily a discretization of sound - but the act of composition, of creating new sounds, requires moving beyond present abstractions into new ones, which most always be done by considering concrete, continuous sound.

Sieves

A sieve, as we'll see in the next section, is a particular way of picking out elements of a discretization with particular attention to the intervals between them - first explored by Iannis Xenakis. Our discussion above will motivate us to explore interesting properties of sieves. More specifically, we will want to explore sieves formally with regards of finding interesting sonorities within specific parameters. This will first entail a process of abstraction and then a re-introduction of our parameters to motivate the properties of sieves we wish to study.

III Quantities

Discretization starts with the peano axiomatic, just as the **natural numbers** $\mathbb{N} = \{1, 2, 3 \dots\}$. Often times it's more useful to consider the **integers** $\mathbb{Z} = \{\dots -2, -1, 0, 1, 2 \dots\}$ since many musical parameters can be both decreased and increased. Note that if we want to increase the spacing of our discretization, we can multiply by an integer m: $m\mathbb{Z} = \{\dots - 2m, -1m, 0, 1m, 2m \dots\}$. We also care about the **integers modulo m**, which we will write as $\mathbb{Z}/m\mathbb{Z}$ which consists of classes of integers under the relationship that $a \sim b$ if and only if $a + nm = b$ for some $n \in \mathbb{Z}$. With these preliminaries out of the way, we can now define sieves.

Musical Sieves

Musical sieves, as defined here, are not precisely the same thing as number theoretic sieves. For the remainder of this paper, it should be assumed that by "sieve" I mean "musical sieve".

Definition 1. **Basic Sieve** is a set $m\mathbb{Z} + k = \{\dots k-2m, k-m, k, k+m, k+2m, k+2m\dots\}$.

Definition 2. A **Basic Cosieve** is a set theoretic complement of a basic sieve $(m\mathbb{Z}+k)^c$ that contains all of the elements of $a \in \mathbb{Z}$ where $a \neq nm + k\ \forall n \in \mathbb{Z}$.

Definition 3. A **Sieve** is a subset of the integers $\{S \subseteq \mathbb{Z}\}$ generated by taking unions or intersections of finitely many basic sieves or cosieves. We will let the space of sieves be called $\mathbb{S}$

Note 4. We could consider more complicated sieves by allowing for infinite unions or intersections. This is an interesting area of exploration but outside the scope of this paper

Note 5. We don't need to consider cosieves as part of this definition. For any basic cosieve we have that

$$(m\mathbb{Z}+k)^c = \bigcup_{j\neq k}^{j\in\{1,2\dots m-1,m\}} (m\mathbb{Z}+j)$$

For example, we have that $(3\mathbb{Z}+0)^c = (3\mathbb{Z}+1) \cup (3\mathbb{Z}+2)$

Definition 6. A **presentation** of a sieve is a specific way of representing it using intersections and unions of basic sieves. We can also define **copresentations** by using basic cosieves instead.

Note 7. A sieve may have multiple presentations, some more complex than others. For example, $\mathbb{Z} = (2\mathbb{Z}+0) \cup (2\mathbb{Z}+1)$. We will want some ways of distinguishing between various presentations.

Definition 8. The **order** of a sieve, $o(S), S \in \mathbb{S}$ is the minimum number of basic sieves necessary to write a presentation of S. By definition we'll take $o() = 0$. We can also define the **co-order**, $o^c(S)$ by taking basic cosieves instead.

Definition 9. The **rank** or period of a basic sieve $r(m\mathbb{Z}+k)$ is m. The rank of a sieve presentation is the least common multiple of the ranks of its constitutive basic sieves. The rank of a sieve $r(S)$ is given by the minimum rank of all of the presentations of S. By considering copresentations we can also consider the corank $r^c(S)$ of a sieve.

Definition 10. We will call sieve of rank m **separated** if it only has presentations of rank m.

Proposition 11. $r(S) = r^c(S)$; $o^c(S)/o(S) \leq r(S)$; $o(S)/o^c(S) \leq r(S)$. To prove this, we note that any basic sieve of rank m can be written as $m-1$ cosieves, and so the co-order of a sieve will always be bounded by m times the order of the sieve. Rank and co-rank must always be equal through this same rewriting.

It may be a little more interesting note, we can write both inequalities above together as $|\log(o(S)) - \log(o^c(S))| \leq \log(r(S))$

Note 12. We can think about $\log(o(S))$ as the *order-complexity* of the sieve S. The thrust of the above proposition is that for low enough rank sieves, any results about order-complexity will roughly hold for co-order, up to the log of the rank. While a basic sieve may be a more natural way of thinking about intervals in most cases - the cosieve view can be thought of classifying the complexity of a work of music by studying its silences instead. We generally expect a complex rhythm to also have a fairly complex set of silences, and the above result shows that this is generally true in the sieve setting. If we're feeling confident, we can form a conjecture (which I have not confirmed is true)

Conjecture 13. Take all $S \in \mathbb{S}$ so that $r(S) = m$, call this set R_m. We conjecture that

$$\lim_{m\to\infty} \sum_{S\in R_m} |\log(o(S)) - \log(o^c(S))| / \log(r(S)) = 0$$

Take for example, a sieve of rank 17, $17\mathbb{Z}$. As a rhythm this is sieve is fairly simple to the listener - we cannot precisely measure the silence, but we'll hear a predictable beat. On the other hand, if tried to write this sieve only using intersections of complements, we would have to take 16 cosieves to do it. If our sieve is a measurement of the only beats where we are silent - if we flip the role of silence and sound in our thought - then the sieve is fairly complex. Rather than hearing a regular beat, we would hear a repeating pulse with a highly irregular silence. This dichotomy can be quantitatively measured by the fact that this sieve has order 1 but co-order 16.

We can create sieves with this dichotomy pretty easily, however, we expect that *most* sieves of order 17 have equally complex silence and co-silence. This conjecture is a precise mathematical statement of that guess.

The monomial model

Changes in notation can bring new insights, in both music and mathematics. Lets create a nicer notation for sieves. Rather than $3\mathbb{Z}+1$ we will write $3x+1$. In general we will write $m\mathbb{Z}+b$ as $mx+b$. We then write $3\mathbb{Z}+1 \cup 3\mathbb{Z}+2$ as $(3x+1) \vee (3x+2)$ We will also write $3\mathbb{Z}+1 \cap 3\mathbb{Z}+2$ as $(3x+1) \wedge (3x+2)$. ($\vee$ stand for 'or'; $\wedge$ stands for 'and'). We will also write $\neg 3x+1$ for $(3\mathbb{Z}+1)^c$. We will call the empty set 0. Note that we need to make $\neg 0$ and $\neg x$ undefined to have well-defined algebraic relations. If we have $mx+b$ with $b \geq m$ we'll take $b \mod m$.

Note that these aren't normal linear expressions: we have expressions like $2x \wedge 2x+1 = x$. We can ask what the algebraic relations are. For listing all of these relations, I'm going to just show one example for clarity, with the general case being implied.

Theorem 14. The algebraic relations on sieves are generated by the following rules (lcm is the least common multiple and gcd is the greatest common divisor):

Logical Rules:

(1) $\neg x = 0$

(2) $\neg 3x = 3x+1 \vee 3x+2$

(3) De Morgan's Laws

(4) Commutativity and Distribution: $p(x) \wedge (q_1(x) \vee q_2(x)) = (p(x) \wedge q_1(x)) \vee (p(x) \wedge q_2(x))$

Partition rules:
(5) $mx \vee mx+1 \vee mx+2 \cdots \vee mx+m-1 = x$

(6) $mx+k \wedge mx+j = 0$ (for $j \neq k$)

(7) $x \wedge p(x) = p(x)$, $0 \vee p(x) = p(x)$, $x \vee p(x) = x$, $0 \wedge p(x) = 0$

Number rules:

(8) $ax+b \vee cx+d = ax/\gcd(a,b)+b \vee cx/\gcd(a,b)+d$

(9) $ax+b \wedge cx+d = \text{lcm}(a,c)x+e$ when $\gcd(a,c)$ divides $b-d$. e is found via euclid's algorithm.

(10) $ax+b \wedge cx+d = 0$ when $\gcd(a,c)$ does not divide $b-d$.

Rules (1)-(4) define all of the logical properties of sieves. Rules (5)-(7) allow us to do reduction, and rules (8) and (9) tell us when our systems of equations given by sieves have solutions. Using these rules it is possible to program a (very slow) algorithm to take a repeating set of numbers and find both its rank-minimal and order-minimal sieve. Xenakis also describes similar rules and an algorithm for sieves, but his algorithm instead focuses on decomposing sieves into prime power intersections.

Proving this theorem is a bit difficult. The logical rules are straightforward, the number rules require some more linear programming to show but are a consequence of the chinese remainder theorem. A full proof would talk about a lot of space and time, so in lieu of that let me instead a show a few results that motivate these rules.

Lemma 15. Take a sequence of numbers $\{a_1, a_2, a_3, \dots\}$ with $a_{k+1} - a_k = a_k - a_{k-1}$ Then $\{a_1, a_2, a_3, \dots\} = mx + b$, a basic sieve.

Proof: Call $a_k - a_{k-1} = l$ then $a_{k+1} = a_k + l = a_{k-1} + 2l = \cdots = a_1 + kl$ so $(a_k - a_{k-1})x + a_1 = \{a_1, a_2, \dots\}$.

Lemma 16. Take any sieve of the form $s_1 \wedge s_2$ where s_1 and s_2 are basic sieves. Assume that the sieve is non-empty, so that $\{a_1, a_2, a_3, \dots\} = s_1 \wedge s_2$. We then have that $a_k - a_{k-1} = a_{k+1} - a_k \forall m$ and so, $s_1 \wedge s_2 = s_3$

Proof: $a_k - a_{k-1}$ is in $ax \wedge cx$ so without loss of generality, we can assume that $s_1 \wedge s_2 = px \wedge qx$

Now a_k is a multiple of p and q if and only if it is a multiple of $\text{lcm}(p,q)$, the least common multiple of p and q. So $ax \wedge cx = \text{lcm}\, a, c$. Alternatively, we could use rule (9).

Theorem 17. Every sieve can be written in the form $s_1 \vee s_2 \vee s_3 \cdots \vee s_n$.

Proof: We can always distribute to write any sieve as

$$(p_{11}(x) \wedge p_{12}(x) \wedge \dots p_{1n}(x)) \vee (p_{21}(x) \wedge \dots p_{2n}(x)) \vee \dots \vee (p_{m1}(x) \wedge p_{m2}(x) \wedge \dots p_{mn}(x)$$

From the above lemma we either write these write any of these $\wedge$ terms as a single basic sieve (including possibly the empty sieve).

Note 18. This tells us that all sieves have a representation as unions of basic sieves. However, note that finiteness is very important for this result - infinite sieves can have intersections that are not reducible to unions.

Lemma 19. Any order-minimal or rank-minimal sieve presentation can be written in the form of the previous theorem.

Proof: The procedure above always reduces the order of a sieve and keeps the rank the same.

Theorem 20. There is only one order minimal presentation of a sieve, up to rearrangement. The presentation is also rank minimal.

Proof: There is only one way to write an order 0 sieve, as 0. By induction, assume the statement is true for sieves up to order $n-1$. We want to show that the statement is true for sieves of order $n+1$.

Assume for contradiction there are two equal order minimal presentations of a sieve of order n and take $s = s_1 \vee s_2 \vee s_3 \dots s_n = r_1 \vee r_2 \vee \dots r_n$. Since these sieves are order minimal, there exist $a_i \in s$ so that $a_i \notin s_j$ where $j \neq i$. Now with possible rearrangement, we must have that $a \in \neg(r_2 \vee r_3 \vee \dots r_n)$ as well. Since by, induction, there is only order minimal sieve of rank n, and with more rearrangement, we must have that $r_2 = s_2, r_3 = s_3 \dots, r_n = s_n$. Therefore $s_1 = r_1$ and there is only one order-minimal presentation of a sieve. This is also rank minimal because any increase in rank must also cause an increase in order.

Transformations

We want to define transformations of sieves so we can understand their symmetries. A transformation, alternatively called *metabolae* by Xenakis, is any map that takes sieves to sieves. A basic transformation takes basic sieves to basic sieves. In general, we will want to take a transformation on sieves and extend it to basic sieves.

Proposition 21. Take a basic transformation, f. For f to be extendable to a general transformation, it is sufficient that $f(s) \wedge f(r) = f(s \wedge r), f(s) \vee f(r) = f(s \vee r), f(\neg s) = \neg f(s)$ for all basic sieves.

Note 22. Not all transformations come from basic transformations. For example, the complement $f(s) = \neg s$ is a well defined transformation but does not come from a basic transformation.

Definition 23. The **identity transformation** acts as identity: $1(s) = s$.

Definition 24. A **transposition**, T_k is a basic transformation that takes $T : mx+b \mapsto mx+b+k$. Transposition extends to general sieves by $x \mapsto x - k$, or more instructively, taking $m\mathbb{Z} + b \mapsto m(\mathbb{Z} - k) + b$

Note 25. Since we defined sieves to be finitely generated from basic sieves, there is only one extension of T_k as a general sieve. And in general every basic transformation only has at most one extension. For infinite sieves, this may not be the case due to godel's incompleteness theorems.

Definition 26. The **conjugation** or **intervallic inversion** transformation $I(s)$ takes $mx + b \mapsto mx - b$. If we think about a sieve as a union of lines, the conjugation operation flips those lines over the x-axis.

Conjecture 27. Let f be any function on the integers so that for any two co-prime numbers, m and n, $f(mn) = f(m)f(n)$. Then a basic transformation $F : mx + b \mapsto f(m)x + b$ extends to a general transformation on all sieves.

Note 28. This is an interesting conjecture but I'm not sure what to do with it musically. Transformations give us an incredibly rich space to talk about musical ideas, but before we get into that we'll need just a little bit more mathematics: we'll need to define some symmetry properties related to transformations.

Definition 29. Let F be any general tranformation of sieves. We call a sieve s. **k-cyclic with respect to F** if applying F k-times to s. A 1-cyclic transformation with respect to F is called **invariant with respect to F**.

Proposition 30. Let s be k-cyclic with respect to F. Then $s \vee F(s) \vee F^2(s) \vee F^3(s) \vee \dots F^{k-1}(s)$ is invariant with respect to F.

Definition 31. Take a general transformation F. Call two sieves equivalent if $s \sim F(s)$. We call the space of sieves modulo this equivalence relation $\mathbb{S}/F$, the F-orbit class of sieves.

IV Measure in music

Now that we've created thorough mathematical tools, we can use them to get to express very interesting musical results. We have at our hands a rich disposal of tools, transformations, complexity measure and viewpoints, let's get to using them!

§**1.** The (T_1, I)-orbit classes for rank m-presentable sieves are pitch class set classes for m divisions of the octave.

§**2.** To write any rank 12-presentable sieve, we can use basic sieves of rank 1,2,3,4,6, and 12. This presents us with an interesting question which sieves of rank 12 are separated? That is to say, just using $/lor$, which ones can only be written using rank 12 basic sieves and not with any lower ones?

§**3.** Well, such a sieve-class cannot have any intervals of size 6, nor can it have 3 adjacent intervals of size 4, 4 adjacent intervals of size 3, nor 6 adjacent intervals of size 2.

By the pigeonhole principle, any such a sieve must be order 5 or lower. There are several of these sieves, but we can look at more interesting properties by narrowing them down.

For example, the only rank 12 separated sieve-class without intervals of size 1 is $12x \vee 12x + 2 \vee 12x + 4 \vee 12x + 7 \vee 12x + 9$ which corresponds to the major pentatonic scale. The other separated sieve-classes of size 5 (forte no. 5-27A/B, forte no. 5-23 A/B, forte no. 5-11 A/B, forte no. 5-2 A/B and forte no. 5-1) with size 1 intervals do not correspond to commonly known scales since step sizes of a semitone are uncommon in five note scales. It is interesting to note that the only separated sieves without z-relations are the major pentatonic and the five note semi-chromatic scale: $12x \vee 12x + 1 \vee 12x + 2 \vee 12x + 3 \vee 12x + 4$.

§**4.** Separated sieves are particularly interesting because they are a property that tracks indecompasibility. All sieves of prime rank are necessarily separated. We can use separated sieves as building blocks for more complicated sieves just as we use prime numbers as building blocks for composite numbers.

Note that any separated sieve of rank m cannot be T_k invariant for $k < m$.

§**5.** As suggested by the monomial model, we can picture any basic sieve as a line over an integer grid, only including the numbers that the line hits in our sieve. This suggests that we can create *continuous sieves* by including integers with an intensity that scales down from the distance from the line. In particular, we could consider a single musical note given as $a_n \exp(-d|\text{round}\,(n-b)/mm + b - n|) \sin n\pi x$. This note would correspond to the sieve $mx + b$, with scaling factor d determining how harshly distance from the sieve should be measured and factor a_n determining how to scale down the notes as n increases.

§**6.** Sieves given by cycles are quite interesting. For example take $12x$. The sieve generated by taking T_2 cycles is $12x \vee 12x + 2 \vee 12x + 4 \vee 12x + 6 \vee 12x + 8 \vee 12x + 10$, which is the whole tone scale. On the other hand, taking T_3 cycles generated by $12x \vee 12x + 1$ gives us $12x \vee 12x + 1 \vee 12x + 3 \vee 12x + 4 \vee 12x + 6 \vee 12x + 7 \vee 12x + 9 \vee 12x + 10$, the octatonic scale. Transpotionally generated sieves can never be separated, for example, the octatonic scale can also be written as $3x \vee 3x + 1$.

§**7.** Another interesting cycle is given by inversion. Take for example, the sieve $3x \vee 4x + 1$. The inversion-generated sieve is $3x \vee 4x + 1 \vee 4x + 3 = 12x \vee 12x + 1 \vee 12x + 3 \vee 12x + 5 \vee 12x + 6 \vee 12x + 7 \vee 12x + 9 \vee 12x + 11$ which is a transposition of messian mode 6. Using inversion equivalence allows us to create simple scales that have very nice symmetry properties. Similar to the above the scale generated by inversion of $3x + 1$ is a transposition of the octatonic scale.

§**8.** Another interesting sieve transformation to explore comes from the indicator function for prime powers is multiplicative and so forms a sieve transformation by acting on the multiple. This allows us to take a 'master sieve' and look at parts of it that only have cycles of a certain prime power. There are other sorts of indicator functions that form well defined sieve operations that can be used as well, such as sets generated by powers of two distinct primes.

For example, we can take the sieve $2x \vee 3x \vee 4x + 1 \vee 5x + 3$. Applying our indicator for prime powers of 2, we can get a sub-sieve with $2x \vee 4x + 2$. In this manner we can also think about building sieves from smaller ones of prime-power orders.

§**9.** Another interesting sieve operation is the doubling shift: $D(mx + b) = 2mx + b + 1$. If we take $3x \vee D(3x) \vee D(D(3x))$ we get $3x \vee 6x + 1 \vee 12x + 2$. Sieves generated this way have very interesting structures, dense down near 0 and less dense higher up. These sorts of sieves may serve as a way to generate new sorts of timbres.

§**10.** We might start to think about making a catalogue of transformations that producing interesting properties for various musical paramters. For example, inversion generates interesting pitch structures, order-minimality creates interesting rhythms, doubling and other infinitary transformations create interesting timbres. It is a bit harder to find transformations that produce interesting sieves for loudness or form, but with a bit more digging there may be a very rich world here to look at.

§**11.** It is interesting to note that sieves and sieve transformations provide generalizations for set classes and microtones in the right contexts. In this sense, studying the transformational properties of sieves provides a new direction to understand other well-explored mathematical structures in music.

§**12.** Repeated applications of sieve transformations also allow us to create structures like sieve dynamical systems and sieve markov chains, where each state corresponds to the application of an additional transformation. There is an immense world of music that could be made in exploring these directions.

V Future Research and Music

There are a number of directions to take this in and I'm excited to explore all of them. First off, I'd want to create nice programming packages that could be used to make sieves and apply sieve transformations that would make music creation much easier. This would be a very interesting project to work on.

Another direction would be to explore down the path of continuous sieves - in fact a musical sieve itself is only one type of a much larger class of algebraic equations over the integers, usually called a diophantine equation. In a very direct way, a sieve is a sonification of a line, or a union of multiple lines. One could sonify any sort of algebraic object in a similar fashion.

A different direction would be to classify all sieves of rank less than 12. I would be surprised if this hasn't been done already - but I could not find any resources doing so. The steps here would be to finish writing my program that can simplify sieve presentations and then use it to write down the order-minimal presentations of different forte numbers.

I do want to create music out of these objects. I've had a lot of enjoyment looking through various sieves and hearing what sorts of sounds they make. I do want to eventually put it all together and make a coherent piece of music out of these ideas.

VI Conclusion

Sieves are robust tools that allow us to explore interesting structures in the discretization of musical parameters. Throughout this paper we've looked at various metrics, models and relationships of sieves and found interesting musical applications of those tools. Sometimes these results can seem too mathematical, but powerful mathematical structures give us a richer language for creative play. Even though notation can often be dense, the power of that notation is that it allows us to think about old objects in new ways. There are lots of new ways to think about sieves, musical avenues still unexplored. I hope this paper can serve as a baseline to help us navigate the manifold opportunities sieves provide.